

PERIODIC REEB ORBITS ON PREQUANTIZATION BUNDLES

PETER ALBERS, JEAN GUTT, AND DORIS HEIN

ABSTRACT. In this paper, we prove that every graphical hypersurface in a prequantization bundle over a symplectic manifold M , pinched between two circle bundles whose ratio of radii is less than $\sqrt{2}$ carries either one short simple periodic orbit or carries at least $\text{cuplength}(M) + 1$ simple periodic Reeb orbits.

1. INTRODUCTION

A *contact form* on a manifold M of dimension $2n - 1$ is a differential 1-form α satisfying $\alpha \wedge (d\alpha)^{n-1} \neq 0$ everywhere. The *Reeb vector field* R_α associated to a contact form α is the unique vector field on M characterized by: $\iota(R_\alpha)d\alpha = 0$ and $\alpha(R_\alpha) = 1$.

In this article we are concerned with prequantization bundles E . That is, E is a \mathbb{C} -bundle over a symplectic manifold (M, ω) with $c_1(E) = -[\omega] \in H^2(M; \mathbb{Z})$. In particular, we assume that the cohomology class $[\omega]$ of the symplectic form admits an integral lift. A Hermitian connection on E gives rise to a connection 1-form α_0 on the corresponding S^1 -bundle Σ over M . The 1-form α_0 is naturally a contact form. Its Reeb vector field is the infinitesimal generator of the S^1 -action on Σ , see [Gei08, Section 7.2] for more details. Moreover, using the Hermitian connection defines circle resp. disk bundles S_R resp. D_R of radius $R > 0$. We will extend α_0 to $E \setminus M$ by pullback.

We call a hypersurface $\Sigma_f \subset E$ *graphical* if it can be written as the graph of a function $f : \Sigma \rightarrow \mathbb{R}_{>0}$ inside E

$$\Sigma_f = \{f(x)x \mid x \in \Sigma\}. \quad (1.1)$$

Then $\alpha_f := f\alpha_0$ is a contact form on Σ_f . We call Σ_f *pinched* between S_{R_1} and S_{R_2} if $\Sigma \subset D_{R_2} \setminus \text{int} D_{R_1}$.

Theorem 1.1. *Let E be prequantization bundle over the symplectic manifold (M^{2n}, ω) . Assume that the graphical hypersurface $\Sigma_f \subset E$ is pinched between S_{R_1} and S_{R_2} with $\frac{R_2}{R_1} < \sqrt{2}$. Then there exists periodic Reeb orbits $\gamma_1, \dots, \gamma_c$ of R_{α_f} with $c = \text{cuplength}(M) + 1$ such that*

$$\pi R_1^2 < \mathcal{A}_{\alpha_f}(\gamma_1) < \dots < \mathcal{A}_{\alpha_f}(\gamma_c) < \pi R_2^2$$

where $\mathcal{A}_{\alpha_f}(\gamma) := \int_\gamma \alpha_f$ is the action or period of a Reeb orbit γ .

We recall the definition of cuplength .

Definition 1.2. Let M be a manifold. The *cuplength* of M is defined as

$$\text{cuplength}(M) := \max \{k \in \mathbb{N} \mid \exists \beta_1, \dots, \beta_k \in H^{\geq 1}(M) \text{ such that } \beta_1 \cup \dots \cup \beta_k \neq 0\}.$$

Corollary 1.3. *In the context of Theorem 1.1, either the minimal period of periodic Reeb orbits of R_{α_f} is less than πR_1^2 or α_f carries at least $\text{cuplength}(M) + 1$ simple periodic Reeb orbits.*

In short, there is either a short periodic orbit or $\text{cuplength}(M) + 1$ simple periodic Reeb orbits.

Remark 1.4. Amongst other theorems, a similar result to Theorem 1.1 has been obtained by Ely Kerman in [Ker16], but with the bound for the number of critical points being $\frac{1}{2} \dim M + 1$. Since the symplectic form is non-degenerate, we have $\text{cuplength}(M) + 1 \geq \frac{1}{2} \dim M + 1$ in general.

As a particular case of Corollary 1.3, we also find the following. We recall that S^{2n-1} is the S^1 -bundle corresponding to a prequantization bundle over \mathbb{CP}^{n-1} and $\text{cuplength}(\mathbb{CP}^{n-1}) = n - 1$.

Corollary 1.5 ([EL80, BLMR85]). *Let S be a hypersurface in \mathbb{R}^{2n} satisfying*

$$\langle \nu_\Sigma(x), x \rangle > r \quad \forall x \in \Sigma, \quad (1.2)$$

where $\nu_\Sigma(x)$ is the exterior unit normal vector of Σ at x . Then S is starshaped and we denote by $\xi = \ker \alpha_0$ the standard contact structure on S . Assume there exists a point $x_0 \in \mathbb{R}^{2n}$ and numbers $0 < r \leq R$ with $R < r\sqrt{2}$ such that:

$$r \leq \|x - x_0\| \leq R \quad \forall x \in \Sigma, \quad (1.3)$$

Assume also that Σ carries at least n geometrically distinct periodic Reeb orbits.

Another proof of this result with the additional assumption that the contact form is non degenerate was given by the second author in [Gut15].

The study of periodic Reeb orbits can be translated in the study of periodic solutions of Hamiltonian systems and has a long history which, probably, started when Poincaré pointed out their interest. The question of lower bounds on the number of simple periodic Reeb orbits on compact manifold is wide open; it is not even known for the standard sphere in \mathbb{R}^{2n} . In fact, the existence of one periodic Reeb orbit on every compact contact manifold (Weinstein conjecture) is still open in dimension greater than 3 where it was proven by Taubes [Tau07]. Taubes result was then improved independantly by Cristofaro-Gardiner and Hutchings [CGH12] and by Ginzburg, Hein, Hryniewicz and Macarini [GHHM13], who proved that every contact form on a closed three-manifold has at least two embedded periodic Reeb orbits.

On the sphere, more is known; Hofer, Wysocki and Zehnder [HWZ95] have shown that on S^3 , every dynamically convex (see [HWZ95]) contact form carries either 2 or infinitely many periodic Reeb orbits. In dimension greater than 3, the conjecture is that any contact form on the $2n - 1$ dimensional sphere defining the standard contact structure admits at least n simple periodic orbits. This conjecture is studied, for instance, in [GG16, LZ02, WHL07, EL80, BLMR85]

For manifolds (of dimension ≥ 5) other than the sphere, very little is known, we refer to [GG16, GK15, AM15, Kan13] for precise statements but we would like to point out that nothing is known outside some restricted class of prequantization bundles.

2. BASIC CONSTRUCTIONS

Let (Σ, α) be a prequantization space over (M, ω) . That is, (M, ω) is a closed connected symplectic manifold with integral symplectic form $[\omega] \in H^2(M, \mathbb{Z})$. We denote by $\wp : \Sigma \rightarrow M$ the principal S^1 -bundle and by $\wp : E \rightarrow M$ the associated complex line bundle with first

Chern class $c_1^E = -[\omega]$. We refer to these bundles as prequantizations spaces. There exists an S^1 -invariant 1-form α on Σ , and hence $E \setminus M$, with the property

$$d\alpha = \wp^* \omega \quad (2.1)$$

which is a contact form on Σ . For more details we refer to [Gei08, Section 7.2]. If we denote by ρ the radial coordinate on E then the 2-form

$$\Omega := d(\pi\rho^2\alpha) + \wp^* \omega = 2\pi\rho d\rho \wedge \alpha + (\pi\rho^2 + 1)\wp^* \omega \quad (2.2)$$

is a symplectic form on E .

In the following we will work on the symplectization $S\Sigma := \Sigma \times \mathbb{R}_{>0}$ of Σ which is equipped with the exact symplectic form $\Omega = d(r\alpha) = dr \wedge \alpha + rd\alpha$. Here r is the natural coordinate on $\mathbb{R}_{>0}$. The coordinate transformation $r = \pi\rho^2$ induces an exact symplectomorphism $(E \setminus M, \pi\rho^2\alpha) \cong (S\Sigma, r\alpha)$. We point out that the Reeb flow θ_t of the Reeb vector field R is 1-periodic due to our convention that $S^1 = \mathbb{R}/\mathbb{Z}$.

Now we fix the basic Hamiltonian functions and study a certain moduli space which will play a crucial role in the proof.

2.1. The Hamiltonian functions and their periodic orbits. In this paper, the initial choice of the Hamiltonian function plays a crucial role. It is defined as a radial function in the complex line bundle $E \rightarrow M$ and has a shape similar to the standard ones in symplectic homology, but eventually becoming constant again.

In order to construct this Hamiltonian function, we first fix a number $R_0 \in \mathbb{R}$ with $1 < R_0 < 2$ and choose constants $A, c \in (0, 1)$, which in addition satisfy

$$c < \frac{R_0 - 1}{1 - \log R_0}, \quad (2.3)$$

$$Ac \left(\exp \frac{R_0 - 1}{c} - 1 \right) < 1.$$

As we also require $c < 1$, this condition is only needed if R_0 is close to 1 as otherwise, the right hand side is larger than 1. The second condition can then be satisfied by choosing A sufficiently small. Then we define the function $k: \mathbb{R}_{>0} \rightarrow \mathbb{R}$ explicitly by the formula

$$k(r) = cr \log r - cr + r(1 - c \log A) + Ac - A. \quad (2.4)$$

Therefore, we have $k(A) = 0, k'(A) = 1$ and

$$|rk''(r)| = c < 1. \quad (2.5)$$

We next set $B = A \exp \frac{R_0 - 1}{c}$. Thus, $A < B \approx Ac$, where \approx becomes an equality in the limit $c \rightarrow 1$ and $R_0 \rightarrow 2$. Moreover, the relations between A, c and R_0 in (2.3) are equivalent to the more readable conditions

$$\log A + 1 < \log R_0 B \quad (2.6)$$

and

$$c(B - A) < 1. \quad (2.7)$$

To define $h: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$, we fix sufficiently small constants $\epsilon, \delta, \bar{\delta} > 0$ and set

$$h(r) = k(r) \quad \text{for } r \in [A - \bar{\delta}, B + \delta] \quad (2.8)$$

and require

$$h'(B + \delta) = R_0 + \epsilon. \quad (2.9)$$

For $r \leq A - \bar{\delta}$, we choose h to be almost linear down to $r = \bar{\delta}$ and then turning to be constant such that $-h(0) \notin [A, A + c(B - A)] + \mathbb{Z}$. This can be achieved by making $\bar{\delta}$ sufficiently

small and keeping the property (2.5). For $r \geq B + \delta$, we choose h to be constant with slope $R_0 + \epsilon < 2$ for some time and then decrease the slope to

$$h'(C) = R_0 \quad \text{at some point } r = C > B. \quad (2.10)$$

After this, we keep slope $R_0 - \epsilon$ for a while until we decrease again to $h'(D) = 1$ for some possibly large $D > C$. By the same pattern, we decrease the slope further to $1 - \epsilon$ for some finite interval before we eventually make $h(r)$ constant for large r . In the non-linear parts, we make all choices such that the condition (2.5) is satisfied, i.e., the slope decreases more slowly as we move further out.

We adjust the various bits of constant slope, $R_0 \pm \epsilon, 1 - \epsilon$, so that the respective values of h at B, C, D are such that the requirements below are met. To sum this up, we construct $h(r)$ such that we get a shape as in see Figure 1 with the following properties:

$$\left\{ \begin{array}{ll} h'(r) \in [0, R_0 + \epsilon] & \text{for all } r \in \mathbb{R}_{>0} \\ \max h'(r) = R_0 + \epsilon & \\ h'(r) = 0 & \iff r \in [0, \bar{\delta}] \text{ or } r \text{ large} \\ h'(r) = 1 & \iff r \in \{A, D\} \\ h'(r) = R_0 & \iff r \in \{B, C\} \\ h(A) = 0, \\ -h(0) \notin [A, A + c(B - A)] + \mathbb{Z} & \\ h(B) = BR_0 - cB + cA - A & \\ CR_0 - h(C) \notin [A, A + c(B - A)] + \mathbb{Z} & \\ D - h(D) \notin [A, A + c(B - A)] + \mathbb{Z} & \\ \lim_{r \rightarrow \infty} h(r) \notin [A, A + c(B - A)] + \mathbb{Z} & \\ h''(r) \geq 0 & \text{for } r \leq R_0 B \\ |rh''(r)| < 1 & \text{for all } r \in \mathbb{R}_{>0} \end{array} \right. \quad (2.11)$$

Note that by our choices of R_0, c and A , the set $[A, A + c(B - A)] + \mathbb{Z}$ is not all of \mathbb{R} as we required $c(B - A) < 1$ and the conditions on $h(0), C, D$ and $\lim_{r \rightarrow \infty} h(r)$ can be satisfied.

Most of these conditions are needed to get a good picture of periodic orbits and their action values. But we point out that the last condition is the most important one as it will enable us to get hold of a certain moduli space (in Step 2 of Theorem 2.2). Using this function $h(r)$, we now define the Hamiltonian function $H: E \rightarrow \mathbb{R}$ simply by

$$H(q) := \begin{cases} h(r) & \text{if } q = (x, r) \in E \setminus M \cong \Sigma \times \mathbb{R}_{>0} \\ h(0) & \text{if } q \in M \end{cases} \quad (2.12)$$

The Hamiltonian function H is smooth since $h(r)$ is constant for $r < \bar{\delta}$.

As a next step, we compute the action values of all 1-periodic orbits for this Hamiltonian function $H = h(r)$. Observe that with our conventions, the Hamiltonian vector field is given by $X_H = h'(r)R$ where R is the Reeb vector field on Σ . Moreover, since the Reeb flow is

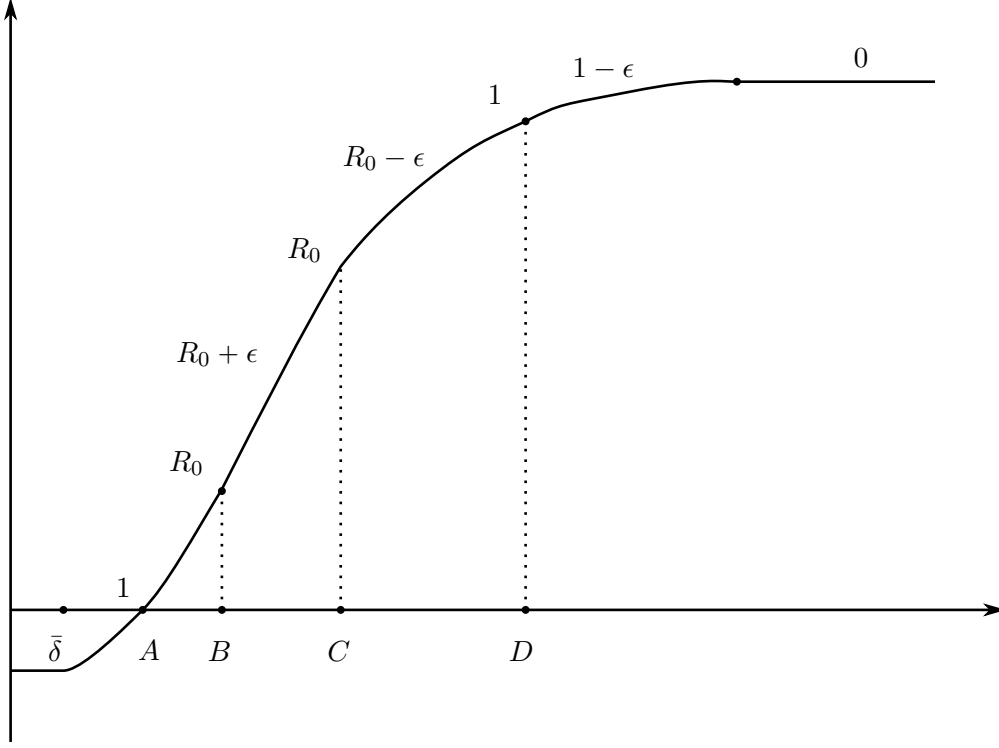


FIGURE 1. The function h . The numbers at the graph indicate the slope at this point / section.

1-periodic the 1-periodic orbits of X_H correspond to values of r with $h'(r) \in \mathbb{Z}$. As we chose $R_0 + \epsilon < 2$ to be the maximal slope of h , the condition $h' \in \mathbb{Z}$ for 1-periodic orbits turns into $h'(r) \in \{0, 1\}$. We get four types of periodic orbits:

- (1) Constant orbits for $r \in [0, \bar{\delta}]$, where h is constant,
- (2) 1-periodic Reeb orbits at $r = A$, where $h'(A) = 1$,
- (3) 1-periodic Reeb orbits at $r = D$, where $h'(D) = 1$ and
- (4) Constant orbits for very large r , where h is again constant.

To compute the action values of these periodic orbits we recall that the Hamiltonian action functional \mathcal{A}_H on (E, Ω) is defined on a covering $\tilde{\Lambda}E$ of the component of contractible loops of the free loop space ΛE of E . This covering has $\frac{\pi_2(E)}{\ker \Omega} \cong \frac{\pi_2(M)}{\ker \omega}$ as deck transformation group. We denote elements by $[\gamma, \bar{\gamma}]$; i.e. concretely γ is a contractible loop and $\bar{\gamma}$ is a disk bounded by γ with the equivalence relation that $(\gamma, \bar{\gamma}) \sim (\gamma', \bar{\gamma}')$ if and only if

$$\begin{cases} \gamma = \gamma' \\ \Omega(\bar{\gamma} \# - \bar{\gamma}') = 0. \end{cases} \quad (2.13)$$

We call $\bar{\gamma}$ a capping of γ . The action functional $\mathcal{A}_H : \tilde{\Lambda}E \rightarrow \mathbb{R}$ is defined by

$$\mathcal{A}_H([\gamma, \bar{\gamma}]) := \int_{D^2} \bar{\gamma}^* \Omega - \int_0^1 H(\gamma(t)) dt. \quad (2.14)$$

An element $[\gamma, \bar{\gamma}]$ is a critical point of \mathcal{A}_H if and only if

$$\gamma'(t) = X_H(\gamma(t)) \quad (2.15)$$

i.e. the 1-periodic orbits of X_H with some capping $\bar{\gamma}$.

As explained above there are 4 types of orbits, either constant orbits or 1-periodic Reeb orbits, each at certain values for r . We point out that all these orbits have natural cappings. For the constant orbits we choose the capping to be a constant disk. For the 1-periodic Reeb orbits we choose the disk in a fiber of E containing the specific Reeb orbit. Using these natural cappings we abbreviate their action values by $\mathcal{A}_H(r)$. Then a simple computation (we recall our convention $S^1 = \mathbb{R}/\mathbb{Z}$) leads to

$$\mathcal{A}_H(r) = rh'(r) - h(r). \quad (2.16)$$

The action values of the critical points with other cappings are obtained by changing the natural cappings by an element in $\frac{\pi_2(E)}{\ker \Omega} \cong \frac{\pi_2(M)}{\ker \omega}$. This changes the action value by an integer since $\omega : \frac{\pi_2(M)}{\ker \omega} \rightarrow \mathbb{Z}$ due to the condition that $[\omega] \in H^2(M; \mathbb{Z})$.

We now compute the action values $\mathcal{A}_H(r)$ for the orbits of different types using the properties of h , see (2.11).

- For the orbits in class (1), our choice of h implies $\mathcal{A}_H(r) = -h(0) \notin [A, A + c(B - A)] + \mathbb{Z}$ for $r \in [0, \bar{\delta}]$.
- For orbits in class (2), we fixed the value of the Hamiltonian function to be zero and therefore get $\mathcal{A}_H(A) = A$.
- For the orbits in class (3), we required $\mathcal{A}_H(D) = Dh'(D) - h(D) \notin [A, A + c(B - A)] + \mathbb{Z}$.
- Finally, for class (4), the condition on $\lim_{r \rightarrow \infty} h(r)$ and that h becomes constant for large r imply that the action value $\mathcal{A}_H(r)$ is not in $[A, A + c(B - A)] + \mathbb{Z}$.

For the second Hamiltonian function L , we consider a rescaled version of h by defining

$$l(r) = h\left(\frac{r}{R_0}\right). \quad (2.17)$$

and define as above $L : E \rightarrow \mathbb{R}$ by

$$L(q) := \begin{cases} l(r) & \text{if } q = (x, r) \in E \setminus M \cong \Sigma \times \mathbb{R}_{>0} \\ l(0) & \text{if } q \in M \end{cases} \quad (2.18)$$

In this case, we have

$$X_L(x, r) = \frac{1}{R_0} h'\left(\frac{r}{R_0}\right) R(x) \quad (2.19)$$

and therefore, to get a periodic orbit, we must have $h'\left(\frac{r}{R_0}\right)$ to be an integer multiple of R_0 . By the conditions on h , we still get the constant periodic orbits as above for $r \in [0, R_0\bar{\delta}]$ and for large r and 1-periodic orbits when $r = R_0B$ and $r = R_0C$.

The action values with the natural cappings are now given by

$$\mathcal{A}_L(r) = rl'(r) - l(r) = \frac{r}{R_0} h'\left(\frac{r}{R_0}\right) - h\left(\frac{r}{R_0}\right). \quad (2.20)$$

Again the action values for the constant orbits are given by the values of h near 0 and near ∞ and therefore are not in $[A, A + c(B - A)] + \mathbb{Z}$. For the orbits at $r = R_0B$, we find

$$\mathcal{A}_L(R_0B) = Bh'(B) - h(B) = BR_0 - h(B) = A + c(B - A). \quad (2.21)$$

Finally, at $r = R_0C$, the properties of h imply that

$$\mathcal{A}_L(R_0C) = Ch'(C) - h(C) \notin [A, A + c(B - A)] + \mathbb{Z}. \quad (2.22)$$

From now on, we restrict our attention to periodic orbits with action in the interval $I = [A, A + c(B - A)]$. These are the orbits at the first time the slope reaches 1, when the Hamiltonian function starts to increase as usually considered in symplectic homology. More concretely, we only have the orbits at $r = A$ for H and at $r = R_0B$ for L with their natural capping being the fiber disk in the complex line bundle E .

2.2. The initial moduli space. We now study the moduli space arising in the continuation homomorphism for a monotone homotopy between the Hamiltonian functions H and L coming from a monotone homotopy between $h(r)$ and $l(r) = h(\frac{r}{R_0})$ constructed above. For this we define

$$h_s(r) = \beta(s)h(r) + (1 - \beta(s))h\left(\frac{r}{R_0}\right), \quad (2.23)$$

where β is a smooth, monotone decreasing cut-off function which is 1 for $s < -1$ and 0 for $s > 0$. Moreover, we require that $\beta'(s) < 0$ for all $s \in (-1, 0)$. Then h_s is a monotone homotopy from h to l . Note that the condition $|rh_s''(r)| < 1$ is still satisfied for all $r > 0$ and $s \in \mathbb{R}$ as for each s the function h_s is a convex combination of h and l which both satisfy the required condition.

Definition 2.1. We fix an almost complex structure J on $S\Sigma$ which is compatible with ω and of SFT-type. This means that $\omega(\cdot, J\cdot)$ defines a Riemannian metric on $S\Sigma$, that $\frac{1}{r}dr = \alpha \circ J$ and that J is invariant under the Liouville flow $(x, r) \mapsto (x, e^t r)$ for $t \in \mathbb{R}$.

As we will use this very explicitly in the technical parts of this paper, we mention here that this implies that $J \frac{\partial}{\partial r} = \frac{1}{r}R$. Using this J , we can now define the moduli space of interest by

$$\mathcal{M} = \left\{ u: S^1 \times \mathbb{R} \rightarrow E \mid \partial_s u + J(\partial_t u - X_s(u)) = 0, \lim_{s \rightarrow \pm\infty} u = \gamma_{\pm} \right\}, \quad (2.24)$$

where X_s is the Hamiltonian vector field for h_s . Moreover, the orbit γ_- is a 1-periodic orbit of X_H at $r = A$ and γ_+ is a 1-periodic orbit of X_L at $r = R_0B$, both having action in the interval $I = [A, c(B - A) - A]$ as computed above. Both Hamiltonian action functionals for H and L are Morse-Bott and the critical manifolds formed by the respective orbits γ_{\pm} are both diffeomorphic to Σ since γ_{\pm} correspond to simple Reeb orbits.

We point out that since all Hamiltonian functions are autonomous, the moduli space \mathcal{M} carries a free S^1 -action given by rotating solutions, $(\tau * u)(s, t) := u(s, t + \tau)$, $\tau \in S^1$. That this action is free follows from considering the asymptotic limits of u . The main result of this section is the following.

Theorem 2.2. *The space \mathcal{M} of solutions u to the Floer equation*

$$\partial_s u + J(\partial_t u - X_s(u)) = 0 \quad (2.25)$$

with

$$\begin{aligned} u(+\infty) &\in \{\gamma_+ \in \text{Crit}\mathcal{A}_L \mid \mathcal{A}_L(\gamma_+) = c(B - A) - A\} \\ u(-\infty) &\in \{\gamma_- \in \text{Crit}\mathcal{A}_H \mid \mathcal{A}_H(\gamma_-) = A\} \end{aligned} \quad (2.26)$$

is compact and carries a free S^1 -action. Moreover, it is S^1 -equivariantly diffeomorphic to Σ

$$\mathcal{M} \cong_{S^1} \Sigma \quad (2.27)$$

and thus

$$\mathcal{M}/S^1 \cong M. \quad (2.28)$$

Finally, all solutions $u \in \mathcal{M}$ are Fredholm regular.

Of course, the statement that \mathcal{M} is compact follows from the rest of the statement as Σ is compact. Therefore, we do not need to prove compactness separately and it suffices to prove $\mathcal{M} \cong \Sigma$.

Even though we will not use it, it is worth pointing out that elements in \mathcal{M} are contributions to the continuation homomorphism between the Floer homologies of H and L .

Before proving this theorem, we give an outline of the proof by mentioning the main steps:

- Step 1 We first show that all elements in \mathcal{M} are contained in a fiber over a closed Reeb orbit γ on Σ of the bundle $E \rightarrow M$ by an energy estimate. In particular, this shows that a solution to the Floer equation in \mathcal{M} can only exist if the asymptotic critical points γ_{\pm} are in the same fiber, i.e., they correspond to the same Reeb orbit γ on Σ .
- Step 2 According to Step 1 we write $u(s, t) = (\gamma(b(s, t)), F(s, t)) \in \mathcal{M}$, where F is the radial coordinate. Then we show that all solutions $u \in \mathcal{M}$ satisfy $b(s, t) = t$ and $F(s, t) = F(s)$ for some function $F: \mathbb{R} \rightarrow \mathbb{R}_{>0}$, i.e. $u(s, t) = (\gamma(t), F(s))$.
- Step 3 According to Step 2 the Floer equation for u reduces to an ODE for F . We prove existence and uniqueness of a solution F for any fixed Reeb orbit γ using the asymptotic conditions at both ends. This completes the proof of $\mathcal{M} \cong_{S^1} \Sigma$ and the equality of the S^1 -actions by rotation in the fiber.
- Step 4 Finally, we prove Fredholm regularity for our solutions.

PROOF. As outlined above, the proof is done in several steps.

Step 1: Let u be an element of the moduli space \mathcal{M} and define $v := \wp(u)$, where $\wp: E \rightarrow M$ is the bundle projection. By our choice of SFT-type J , the projection \wp is holomorphic with respect to the complex structures J on E and j on M . As the Reeb vector field and thus also the Hamiltonian vector field X_s always point in fiber direction we have $\wp_* X_s = 0$. Therefore, v solves the unperturbed Cauchy-Riemann equation

$$\partial_s v + i\partial_t v = 0. \quad (2.29)$$

Using $\wp^* \omega = d\alpha$ and that $\gamma_{\pm} := \lim_{s \rightarrow \pm\infty} u$ satisfies

$$\gamma'_-(t) = \underbrace{h'(A)}_{=1} R(\gamma_-(t)) \quad \text{and} \quad \gamma'_+(t) = \frac{1}{R_0} \underbrace{h'(B)}_{=R_0} R(\gamma_+(t)), \quad (2.30)$$

see (2.19), we compute the energy of v .

$$\begin{aligned} E(v) &= \int_{S^1 \times \mathbb{R}} v^* \omega \\ &= \int_{S^1 \times \mathbb{R}} u^* \wp^* \omega \\ &= \int_{S^1 \times \mathbb{R}} u^* d\alpha \\ &= \int_{S^1} \gamma_+^* \alpha - \int_{S^1} \gamma_-^* \alpha \\ &= 1 - 1 = 0. \end{aligned} \quad (2.31)$$

This shows that v is constant since it is holomorphic and has vanishing energy. Therefore, u is contained in the fiber over this constant. This completes the proof of Step 1.

Step 2: Since every element in \mathcal{M} is contained entirely in a fiber of $\wp : E \rightarrow M$ we can use the Reeb direction and the radial direction as a coordinate system. Thus, we can write an element $u \in \mathcal{M}$ as

$$u(s, t) = (\gamma(b(s, t)), F(s, t)), \quad (2.32)$$

where γ is the Reeb orbit in that fiber and F denotes the radial component. The Floer equation (2.25) in these coordinates becomes a system of PDEs for F and b

$$\begin{aligned} \partial_s b + \frac{1}{F} \partial_t F &= 0 \\ \partial_s F - F \partial_t b + F h'_s(F) &= 0. \end{aligned} \quad (2.33)$$

Setting $G = \log F$ and dividing the second equation by F , this turns into

$$\begin{aligned} \partial_s b + \partial_t G &= 0 \\ \partial_s G - \partial_t b + h'_s(e^G) &= 0. \end{aligned} \quad (2.34)$$

The following argument is based on an argument by Salamon-Zehnder from [SZ92]. We are grateful to W. Merry for pointing us to the article [BO09] by Bourgeois-Oancea who use [SZ92] in a similar fashion.

We linearize equation (2.34) in t -direction and set $\zeta = (\zeta_1, \zeta_2) := (\partial_t b, \partial_t G)$. Then ζ solves the linear system

$$\begin{aligned} \partial_s \zeta_1 + \partial_t \zeta_2 &= 0 \\ \partial_s \zeta_2 - \partial_t \zeta_1 + e^G h''_s(e^G) \zeta_2 &= 0. \end{aligned} \quad (2.35)$$

Combining these two equations into a vector equation gives

$$\partial_s \zeta + J \partial_t \zeta + \begin{pmatrix} 0 & 0 \\ 0 & F h''_s(F) \end{pmatrix} \zeta = 0. \quad (2.36)$$

To this equation we can apply [SZ92, Prop. 4.2] as the matrix norm is $\|F h''_s(F)\| < 1$ by our choice of the Hamiltonian function, namely $|r h''_s(r)| < 1$ for all r . [SZ92, Prop. 4.2] asserts that ζ is independent of t . Thus (2.35) simplifies to

$$\begin{aligned} \partial_t \zeta_1 &= 0 \\ \partial_t \zeta_2 &= 0 \\ \partial_s \zeta_1 &= 0 \\ \partial_s \zeta_2 + F h''_s(F) \zeta_2 &= 0. \end{aligned} \quad (2.37)$$

The first three equations imply that ζ_1 is constant and ζ_2 is independent of t . Since H is a Morse-Bott Hamiltonian, the asymptotic periodic orbit $\lim_{s \rightarrow -\infty} u =: \gamma_-$ sits in a critical manifold diffeomorphic to Σ , see above. In particular, the Morse-Bott property implies exponential convergence of $u = (\gamma, F)$ and all its derivatives to γ_- in normal direction. The normal direction coincides here with the radial direction. In other words $F(s)$ converges exponentially fast to $r = A$ and all its derivatives converge exponentially fast to 0. Therefore $\zeta_2 = \partial_t G = \frac{1}{F} \partial_t F$ also converges to 0, that is

$$\lim_{s \rightarrow -\infty} \zeta_2 = 0. \quad (2.38)$$

Since ζ_2 is independent of t the last equation in (2.37) is now an ODE for $\zeta_2(s)$. For $s \rightarrow -\infty$ the coefficient in the 0-th order term $Fh_s''(F)$ becomes s -independent and converges to $Ah''(A) = c \in (0, 1)$, see (2.5). Therefore, asymptotically, we have

$$\zeta_2 \sim e^{-cs} \quad \text{as } s \rightarrow -\infty. \quad (2.39)$$

Together with the vanishing asymptotic condition for ζ , this implies that $\zeta_2 \equiv 0$.

Going back to the original equation in b and G , we now have found that $\partial_t G = \zeta_2 = 0$. This shows that G , and therefore $F = e^G$, is independent of t .

For b , we now use the first equation in (2.34) to find that $\partial_s b = 0$. By the above argument, we know that $\partial_t b = \zeta_1 = \text{const}$ and therefore, we have

$$b(s, t) = \text{const} \cdot t + b(0). \quad (2.40)$$

As u converges to the Reeb orbit $\gamma(t)$, this asymptotic condition implies that

$$b(s, t) = t \quad \forall t. \quad (2.41)$$

This completes the proof of Step 2. The details of F will be studied in Step 3.

Step 3: In this step, we prove existence and uniqueness of Floer trajectories in the moduli space \mathcal{M} in the fiber over a given Reeb orbit γ . Step 2 reduces the Floer equation (2.25), see also (2.33), to a 1-dimensional ODE for F

$$\begin{aligned} \partial_s F &= -F(h'_s(F) - 1) \\ \lim_{s \rightarrow -\infty} F(s) &= A \\ \lim_{s \rightarrow \infty} F(s) &= BR_0. \end{aligned} \quad (2.42)$$

We want to show existence and uniqueness for F . For this we use a phase space analysis at the boundaries.

We note that for $s < -1$, the function $h_s(r) = h(r)$ is independent of s and that $h'(r)$ is increasing on the interval $r \leq B + \epsilon$ with $h'(A) = 1$. Therefore, for $s < -1$, the function $F(s) \equiv A$ is a solution.

Now we show that no other function solves the ODE problem for $s < -1$. By the asymptotic condition at $s = -\infty$, the function $F(s)$ is less than B for some $s_0 < -1$ since $A < B$.

If $F(s_0) < A$, then $h'(s_0) < 1$ and therefore, the coefficient of F in (2.42) is negative. This shows that F is increasing in s . In turn, as s decreases, $F(s)$ decreases further and further and this contradicts the asymptotic condition $\lim_{s \rightarrow -\infty} F(s) = A$.

Similarly, if $F(s_0) > A$, then $h'(s_0) > 1$ and therefore, the coefficient of F in (2.42) is positive. This shows that F is decreasing in s . Again in turn, as s decreases, $F(s)$ is increasing and this contradicts again the asymptotic condition $\lim_{s \rightarrow -\infty} F(s) = A$.

Combined, this shows that for $s < -1$, the only solution satisfying the asymptotic condition at $-\infty$ is the constant solution $F(s) \equiv A$. Therefore, we can turn (2.42) into an initial value problem. In particular, there is a unique maximal solution to the ODE with asymptotic condition $\lim_{s \rightarrow -\infty} F(s) = A$. It remains to check that this maximal solution is defined on \mathbb{R} and satisfies the asymptotic condition as $s \rightarrow \infty$.

For this we first switch again to $G = \log F$. The ODE for G is then

$$G'(s) = 1 - h'_s(e^G) \quad (2.43)$$

and we have $G(s) = \log A$ for $s \leq -1$. By construction of the Hamiltonian function, $0 \leq h'_s(r) < 2$ for all s and r . In particular, the maximal solution is defined on \mathbb{R} .

From here on, we can again use a phase space analysis for the behavior for $s \geq 0$, where $h_s(r) = l(r)$. Our choice of c and B imply that we have

$$G(0) \leq \log A + 1 < \log R_0 B. \quad (2.44)$$

Therefore, we have $F(0) < R_0 B$ and therefore, we have $h'_s(F(0)) < 1$. This implies that F is increasing and therefore converging to the next value where $h'(r) = 1$ which is $r = R_0 B$. This proves the desired asymptotic behavior of our solution and therefore existence and uniqueness of a Floer trajectory in every fiber and therefore also that $\mathcal{M} \cong \Sigma$ and compactness of \mathcal{M} .

Step 4: It remains to prove Fredholm regularity of the Floer trajectories $u(s, t) = (\gamma(t), F(s))$ studied above; i.e. we need to show that the Fredholm operator given by the linearized Floer equation is surjective. The linearized Floer equation for a vector field $X(s, t)$ along our special solution $u(s, t) = (\gamma(t), F(s))$ is

$$\nabla_s X + J \nabla_t X + \nabla_X (J \partial_t u - \nabla h_s \circ u) = 0 \quad (2.45)$$

with $h_s = \beta(s)h(r) + (1 - \beta(s))h_s(\frac{r}{R_0})$. To prove that this Fredholm operator is surjective, we consider injectivity of the adjoint operator which leads to the equation

$$\nabla_s X - J \nabla_t X - \nabla_X (J \partial_t u - \nabla h_s \circ u) = 0. \quad (2.46)$$

That is, we need to show that all solutions X with vanishing boundary conditions are constant equal to 0.

We set $x = e^r$ and use again the function $G(s) = \log F(s)$. Splitting coordinates, we write $X = X_1(s, t)R + X_2(s, t)\frac{\partial}{\partial x} + X_3$, where X_3 is contained in the contact distribution ξ . As we already know that the image of every solution is contained in a fiber, we consider again the projection $\varphi : E \rightarrow M$. That is, $v = \varphi(u)$ is constant and solves an unperturbed Cauchy-Riemann equation. Since $\varphi_* : \xi \rightarrow TM$ is fiberwise a symplectic isomorphism it is enough to show that $\varphi_* X_3$ vanishes. The latter solves the linearized Cauchy-Riemann equation in a single fiber, namely $T_v M$. Therefore, the boundary conditions together with the maximum principle show that $\varphi_* X_3 = 0$.

Next we compute the radial and the Reeb direction of the linearized equation for each term in $X = X_1(s, t)R + X_2(s, t)\frac{\partial}{\partial x}$ separately. To do this, we use the special form of our solution to observe

$$\nabla_R = \frac{\partial}{\partial t}, \quad G' e^G \nabla \frac{\partial}{\partial x} = \frac{\partial}{\partial s}, \quad \dot{\gamma} = R \quad (2.47)$$

where we choose the standard Riemannian metric in the coordinates (θ, x) on the radial cylinder $S^1 \times \mathbb{R}$. This cylinder corresponds to the fiber over the given Reeb orbit.

For $X_1 R$, the left hand side becomes

$$\partial_s X_1 R + e^{G(s)} \partial_t X_1 \frac{\partial}{\partial x} \quad (2.48)$$

and for $X_2 \frac{\partial}{\partial x}$, we get

$$\partial_s X_2 \frac{\partial}{\partial x} - \frac{1}{e^G} \partial_t X_2 R - X_2 \left(1 + \frac{\partial_s h'_s}{G' e^G} \right) \frac{\partial}{\partial x}. \quad (2.49)$$

The next thing to show is that dividing by G' in the last term is not an issue. For $s \leq -1$, we have shown that G is constant and therefore, $G'(s) = 0$. But in this area, $h_s(e^G)$ is also constant and therefore, all covariant derivatives of ∇h_s also vanish. In particular, this

fractional term vanishes in this area. Moreover, this term is smooth globally since it is the covariant derivative of something smooth.

We now show that for $s > -1$, we have $G' \neq 0$, so that we can safely keep the division by G' in our formulas keeping in mind that this term vanishes, whenever the division by zero would present a problem.

To show this, observe that by the argument in Step 3, the function G is increasing for $s > 0$ and we therefore only need to exclude $G' = 0$ on the interval $(-1, 0)$. We denote by $\rho(s)$ the function $\rho: \mathbb{R} \rightarrow (0, R_0 B]$ defined by the property

$$h'_s(\rho(s)) = 1. \quad (2.50)$$

Then we have

$$\rho(-1) = A, \quad \rho(0) = R_0 B \quad (2.51)$$

as these are the point of slope 1 for $h = h_s$ for $s \leq -1$ and $l = h_s$ for $s \geq 0$. Observe that $e^{G(s)} = \rho(s)$ is equivalent to $G'(s) = 0$ by the ODE for G . Our goal is to show that $e^G < \rho$ for all $s \in (-1, 0)$, which excludes vanishing of G' in this area.

Assume this is not the case and there is some $s_0 \in (-1, 0)$ such that $e^{G(s_0)} \geq \rho(s_0)$, which means $G'(s_0) \leq 0$. We analyze G'' at s_0 and compute

$$G''(s_0) = -h''_{s_0}(e^{G(s_0)})e^{G(s_0)}G'(s_0) - \partial_s h'_{s_0}(e^{G(s_0)}). \quad (2.52)$$

As we know that $e^G < R_0 B$, the first term is non-negative at s_0 as $h''_s(e^{G(s_0)})e^{G(s_0)}$ is positive and $G'(s_0) \leq 0$. For the second term, we have

$$h_s(r) = h\left(\frac{r}{R_0}\right) + \beta(s)\left(h(r) - h\left(\frac{r}{R_0}\right)\right). \quad (2.53)$$

Therefore, we get

$$\partial_s h'_s(r) = \beta'(s)\left(h'(r) - \frac{1}{R_0}h'\left(\frac{r}{R_0}\right)\right) \quad (2.54)$$

We are interested in the sign of this on the interval $(-1, 0)$. Again we analyze this term by term. By our choice of β , we have $\beta'(s)$ is strictly less than zero on this interval. The choice of c and B imply that G stays in the area where $h'' \geq 0$. Therefore, we have $h'\left(\frac{r}{R_0}\right) \leq h'(r)$.

The condition that $R_0 > 1$ now implies that also $\frac{1}{R_0}h'\left(\frac{r}{R_0}\right) < h'(r)$ and therefore, we indeed have

$$\partial_s h'_s(r) < 0. \quad (2.55)$$

This gives the strict inequality $G''(s) > 0$ if $s \in (-1, 0)$ and $G'(s) \leq 0$. In particular, we have shown that $G''(s_0) > 0$. Therefore, for some $s_1 < s_0$ we have $G'(s_1) < G'(s_0) \leq 0$. As above, this implies that we now have the strict inequality $h'_{s_1}(e^{G(s_1)}) > 1$, which implies that

$$e^{G(s_1)} > \rho(s_1). \quad (2.56)$$

Now the above argument with strict inequalities shows that G' is increasing in s . As s goes from s_1 to -1 (i.e. negative s -direction), this means that G' is getting more negative, i.e., G is increasing faster. Therefore, in negative s -direction, the function e^G is therefore increasing but ρ is decreasing. In particular, we have $G(s) > \rho(s)$ for all $s < s_0$ and the gap between e^G and ρ get larger.

But the asymptotic values of e^G and ρ are both A . This contradiction shows that

$$e^{G(s)} < \rho(s) \text{ for all } s \in (-1, 0). \quad (2.57)$$

In particular, we have $G'(s) \neq 0$ on $(-1, 0)$ and therefore do not need to worry about the division by G' in the part of the linearized Floer equation coming from (2.49).

Now we can go back to studying the linearized Floer equation (2.46) along our solutions. The left hand side of the equation is the sum of the expressions in (2.48) and (2.49) and if we sort by direction, i.e., collect the coefficients of R and of $\frac{\partial}{\partial x}$, the conditions on X_1 and X_2 are given by the system of equations

$$\begin{aligned} 0 &= \partial_s X_1 - \frac{\partial_t X_2}{e^G} \\ 0 &= \partial_s X_2 + e^G \partial_t X_1 + X_2 \left(1 + \frac{\partial_s h'_s}{G' e^G} \right). \end{aligned} \quad (2.58)$$

The first step to show that this implies $X_1 = 0 = X_2$, we multiply the first equation by e^G and differentiate with respect to s to find

$$0 = G' e^G \partial_s X_1 + e^G \partial_{ss} X_1 - \partial_{st} X_2 \quad (2.59)$$

and therefore

$$\partial_{st} X_2 = G' e^G \partial_s X_1 + e^G \partial_{ss} X_1 \quad \text{and} \quad \partial_t X_2 = e^G \partial_s X_1. \quad (2.60)$$

Now we differentiate the second equation in (2.58) with respect to t and get

$$0 = \partial_{st} X_2 + e^G \partial_{tt} X_1 + \partial_t X_2 \left(1 + \frac{\partial_s h'_s}{G' e^G} \right) \quad (2.61)$$

as all other terms are independent of t . Substituting the X_2 -terms using (2.60), we get

$$\begin{aligned} 0 &= G' e^G \partial_s X_1 + e^G \partial_{ss} X_1 + e^G \partial_{tt} X_1 + e^G \partial_s X_1 \left(1 + \frac{\partial_s h'_s}{G' e^G} \right) \\ \Leftrightarrow \quad 0 &= \partial_{ss} X_1 + \partial_{tt} X_1 + \partial_s X_1 \left(G' + 1 + \frac{\partial_s h'_s}{G' e^G} \right). \end{aligned} \quad (2.62)$$

This is a linear PDE for X_1 and we will use the maximum principle to show that the vanishing boundary conditions at $s = \pm\infty$ imply that X_1 is constant equal to zero. To do this, we use Theorem 3.1 in [GT83] as we only have terms involving derivatives of X_1 . The main step is to transform the cylinder into a bounded domain in \mathbb{R}^2 and then check ellipticity in this setting. Once this is done, the maximum principle shows that X_1 takes maximum and minimum on the boundary, where we know $X_1 = 0$ and therefore, this shows $X_1 = 0$ everywhere.

The change of coordinates we consider is first $(\sigma, \tau) = (\operatorname{arccot} s, 2\pi t)$ identifying the infinite cylinder with the punctured open disk of radius π and then changing to cartesian coordinates $(x, y) = (\sigma \cos \tau, \sigma \sin \tau)$. Then the boundary conditions on the cylinder are reflected by the requirement that $X_1(0) = 0$ and $X_1 = 0$ on the boundary of the disk of radius π .

As the theorem only requires ellipticity, it suffices to change the coordinates for the second order terms as the coordinate changes will not introduce terms of order zero and the first order terms do not affect ellipticity. These coordinate changes applied to (2.62) while keeping the notation $\sigma = \sqrt{x^2 + y^2}$ give the following second order terms in cartesian coordinates

$$x^2 \left(1 + \frac{\sin^2 \sigma}{\sigma^2} \right) \partial_{xx} + y^2 \left(1 + \frac{\sin^2 \sigma}{\sigma^2} \right) \partial_{yy} + 2xy \left(\frac{\sin^2 \sigma}{\sigma^2} - 4\pi^2 \right) \partial_{xy}. \quad (2.63)$$

It is easy to check that this is elliptic by writing the coefficients into a matrix and checking that it is positive definite by checking that the leading principal minors are all positive. Then

Theorem 3.1 from [GT83] applies and the vanishing boundary conditions imply that X_1 is indeed constant equal to zero.

Now we turn to X_2 , where we first observe that by the first equation in (2.58), the constant vanishing of X_1 implies that X_2 is independent of t . The middle term vanishes in the second equation as $X_1 = 0$ and we get the new equation

$$\partial_s X_2(s) = -X_2(s) \left(1 + \frac{\partial_s h'_s}{G' e^G} \right) \quad (2.64)$$

for X_2 , which is equivalent to

$$\partial_s \log X_2 = -1 - \frac{\partial_s h'_s}{G' e^G}. \quad (2.65)$$

By simple integration, we can compute X_2 explicitly and find

$$X_2 = a \cdot e^{\int -1 - \frac{\partial_s h'_s}{G' e^G} ds}, \quad (2.66)$$

where a is some constant which is determined by the boundary conditions. We consider the asymptotic behavior as $s \rightarrow -\infty$.

We again point out that as h_s is smooth, the same is true for its derivatives and by construction, the fractional term in the integral comes from the covariant derivative of ∇h_s at the point $u(s, t)$. For $s < -1$, u is independent of s and this term is constant. Thus the covariant derivative vanishes for such s . Therefore, the fraction in the exponent vanishes for $s \leq -1$ and the equation becomes

$$X_2 = a e^{-s}. \quad (2.67)$$

But as the boundary conditions of u are fixed, X_2 necessarily vanishes at $s = -\infty$ which implies $a = 0$. We therefore have $X_2(s) = 0$ for all $s \in \mathbb{R}$, which implies that $X_2(s, t) = 0$ as required.

This establishes Fredholm regularity of the unique Floer trajectory in each fiber and completes the proof of Theorem 2.2 \square

2.3. The pinched contact form. We now consider a contact form on Σ induced by the embedding of Σ as graph of a function $f: \Sigma \rightarrow \mathbb{R}_{>0}$ in the complex line bundle. That the contact form is pinched between two multiples of the standard contact form above is reflected by the condition

$$1 \leq f(x) \leq R_0 \quad (2.68)$$

for all $x \in \Sigma$. The function h above is monotone increasing and therefore, this pinching condition also implies that

$$h(r) \geq h\left(\frac{r}{f(x)}\right) \geq h\left(\frac{r}{R_0}\right). \quad (2.69)$$

We now want to define a homotopy from $h(r)$ to $h(\frac{r}{R_0})$ that is not strictly radial as above, but passes through $h(\frac{r}{f(x)})$ instead.

Before that we recall that the cuplength of M is defined as

$$\text{cuplength}(M) := \max \{k \in \mathbb{N} \mid \exists \beta_1, \dots, \beta_k \in H^{\geq 1}(M), \beta_1 \cup \dots \cup \beta_k \neq 0\} \quad (2.70)$$

and denote $k := \text{cuplength}(M)$.

Similar to the function β above, we now define three functions $\beta_1, \beta_2^\rho, \beta_3^\rho: \mathbb{R} \rightarrow [0, 1]$ depending smoothly on a parameter $\rho > 0$ such that

$$\begin{aligned}\beta_1(s) + \beta_2^\rho(s) + \beta_3(s) &= 1 \quad \forall s \in \mathbb{R} \\ \beta_1(s) &\equiv 1 \quad \forall s \leq -1 \\ \beta_2^\rho(s) &\equiv 1 \quad \forall s \in [0, \rho k] \\ \beta_3^\rho(s) &\equiv 1 \quad \forall s \geq \rho k + 1.\end{aligned}\tag{2.71}$$

Furthermore, we require β_1 to be monotone decreasing and β_3^ρ to be monotone increasing. For $\rho = 0$, we choose $\beta_2^0 \equiv 0$ and $\beta_1 = \beta$, where β is the function used above for the radial homotopy and β_i depend smoothly on ρ . Furthermore, we require the convergence as $\rho \rightarrow 0$ to be a C_{loc}^∞ -convergence of β_i^ρ to the specified functions β_i^0 .

Now consider the homotopy

$$H_s^\rho(x, r) = \beta_1(s)h(r) + \beta_2^\rho(s)h\left(\frac{r}{f(x)}\right) + \beta_3^\rho(s)\left(\frac{r}{R_0}\right).\tag{2.72}$$

The pinching condition (2.69) implies that with this choice, we have

$$\frac{\partial H_s^\rho}{\partial s} \leq 0.\tag{2.73}$$

Therefore, the action estimate

$$E(u) \leq \mathcal{A}(u(+\infty)) - \mathcal{A}(u(-\infty))\tag{2.74}$$

holds for all solutions to the Floer equation

$$\partial_s u + J(\partial_t u - X_s^\rho(u)) = 0\tag{2.75}$$

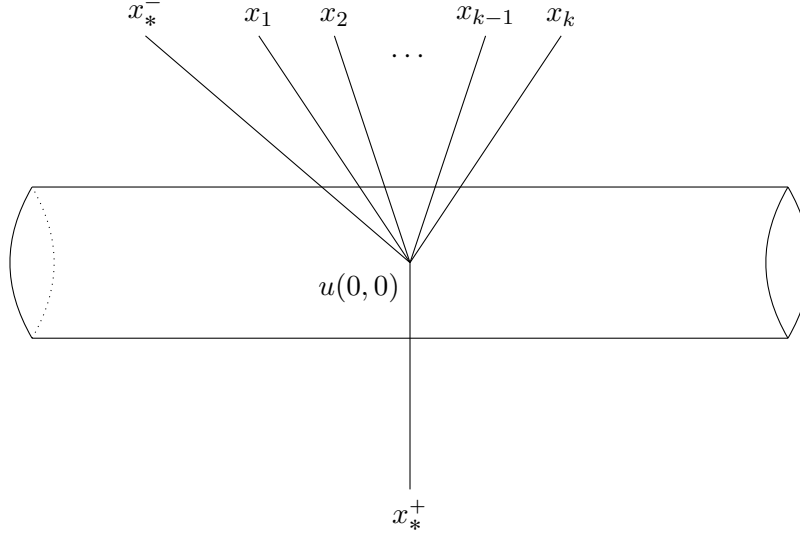
with finite energy, where X_s^ρ is now the Hamiltonian vector field of H_s^ρ .

3. PROOF OF THEOREM 1.1

Now we are in a position to prove our main theorem. The idea is inspired by previous work of two of the authors in [AH16], which in turn is based on the first authors work in [AM10]. To this end, we assume that the Hamiltonian vector field of $h\left(\frac{r}{f(x)}\right)$ has only finitely many periodic orbits. Otherwise, there are infinitely many critical points of the action functional and there is nothing to prove.

The only essential difference to [AH16] is that we do not assume that the symplectic manifold (E, Ω) has any kind of nice behavior concerning bubbling off of holomorphic sphere. In fact, even though (M, ω) satisfies $[\omega] \in H^2(M; \mathbb{Z})$ the manifold (E, Ω) is not necessarily semi-positive. We will rule out bubbling-off of holomorphic spheres by a simple energy argument instead. The non-compactness of E poses no problem, since it is convex at infinity.

3.1. Defining the moduli spaces. Let us describe the setup from [AH16]. Choose generic Morse functions f_*, f_1, \dots, f_k on M such that there are critical points x_*^\pm of f_* and x_i of f_i for $i = 1, \dots, k$ corresponding to cohomology classes whose product is non-zero. We refer to [Sch93] for details on the Morse theoretic cup-product and note here only that the stable manifolds of x_1, \dots, x_k have non-empty intersection, i.e., there are Morse trajectories η_i converging to x_i such that all $\eta_i(0)$ agree and $\eta_i(0) \in W^u(x_*^-, f_*) \cap W^s(x_*^+, f_*)$. We lift these Morse functions to S^1 -invariant Morse-Bott functions on Σ . Then we extend these lifts S^1 -invariantly to Morse-Bott functions on E such that near $\Sigma \times \{A\}$, the extensions have the

FIGURE 2. The moduli space at $\rho = 0$

form $f_i + q_i$ and $f_* + q$, where q_i, q are positive definite quadratic forms in the radial direction. Abusing notation, we also denote these extensions by f_i and f_* as it will always be clear from the context whether we work on E or on M .

We add a marked point to the S^1 -family of lifts of the curves η_i to single out again a bouquet of curves in E as in 2. Again, we abuse notation for these curves in the lifts of the η_i and denote them also by η_i .

We now define the moduli space

$$\widehat{\mathfrak{M}} := \left\{ (\rho, u) \left| \begin{array}{l} \rho \geq 0, u = (\gamma, F) \text{ solves (2.75),} \\ E(u) < \infty, F(-\infty) = A, F(+\infty) = R_0 B \end{array} \right. \right\}. \quad (3.1)$$

At the boundary for $\rho = 0$, we have

$$\partial \widehat{\mathfrak{M}} := \widehat{\mathfrak{M}}|_{\rho=0} = \mathcal{M} \cong \Sigma, \quad (3.2)$$

where \mathcal{M} is the moduli space studied above. Indeed, in this case, H_s^0 is the same homotopy of Hamiltonian functions h_s as before, where we have established Fredholm regularity for this moduli space.

We now add the bouquet of gradient flow lines to the picture and define the moduli space of interest for the proof

$$\mathfrak{M} := \left\{ (\rho, u) \in \widehat{\mathfrak{M}} \left| \begin{array}{l} u(0,0) \in W^u(x_*^-, f_*), u((k+1)\rho, 0) \in W^s(x_*^+, f_*) \\ u(i\rho, 0) = \eta_i(0), i = 1, \dots, k \end{array} \right. \right\}. \quad (3.3)$$

We first study the boundary of this moduli space at $\rho = 0$, where the cylinders can be viewed as elements of \mathcal{M} studied above. The knowledge of \mathcal{M} immediately gives the first insight for $\mathfrak{M}|_{\rho=0}$. Note that we identify $\widehat{\mathfrak{M}}$ at the boundary $\rho = 0$ with Σ and the cylinders $u(s, t)$ are contained in the fibers of the S^1 -bundle over M . On the quotient $M = \Sigma/S^1$, the cylinder thus projects to a point and we have a standard Morse bouquet as usual in the Morse theoretic cup-product.

Proposition 3.1. *The moduli space $\partial\mathfrak{M} = \mathfrak{M}|_{\rho=0}$ is Fredholm regular for $\rho = 0$.*

PROOF. By Theorem 2.2, $\mathcal{M} = \widehat{\mathfrak{M}}|_{\rho=0}$ consists of Fredholm regular solutions to the Floer equation (2.25) and is isomorphic to Σ . As $\rho = 0$ and all $\eta_i(0)$ agree by construction of the Morse bouquet, the condition $u(0,0) = \eta_i(0)$ fixes the point in Σ that u should go through. This singles out exactly one solution to the Floer equation, which is Fredholm regular. The evaluation map

$$ev : \mathcal{M} \rightarrow \Sigma \times \{A\} : u \mapsto u(0,0) = (\gamma(0), F(0)) = (\gamma(0), A)$$

is clearly a submersion according to the identification of \mathcal{M} with Σ in Theorem 2.2. Combining this with the Fredholm regularity of $\partial\widehat{\mathfrak{M}}$, we have shown that $\partial\mathfrak{M}$ is Fredholm regular by Theorem 2.2 above. \square

The main point of the proof is breaking of Floer trajectories in certain limits. For this, we need to show the following

Proposition 3.2. *There exists a sequence $(\rho_n, u_n) \in \mathfrak{M}$ such that $\rho_n \rightarrow \infty$.*

PROOF. Assume that this is not the case and for all sequences $\{(\rho_n, u_n)\}_{n \in \mathbb{N}}$ in \mathfrak{M} , the parameter ρ stays bounded, i.e., we have

$$\sup_{n \in \mathbb{N}} |\rho_n| < \infty.$$

We first show that in this case, the moduli space is compact. Let $\{(\rho_n, u_n)\}_{n \in \mathbb{N}}$ be a sequence in \mathfrak{M} . Possibly by passing to a subsequence, we can assume ρ_n converges to ρ^* . We would like to apply a result by Schwarz in [Sch95, Proposition 4.3.11] stating that convergence of Floer trajectories in C_{loc}^∞ without breaking or bubbling already implies convergence in $H^{1,p}$. We already have the C_{loc}^∞ -convergence and thus need only to show that there is no bubbling nor breaking.

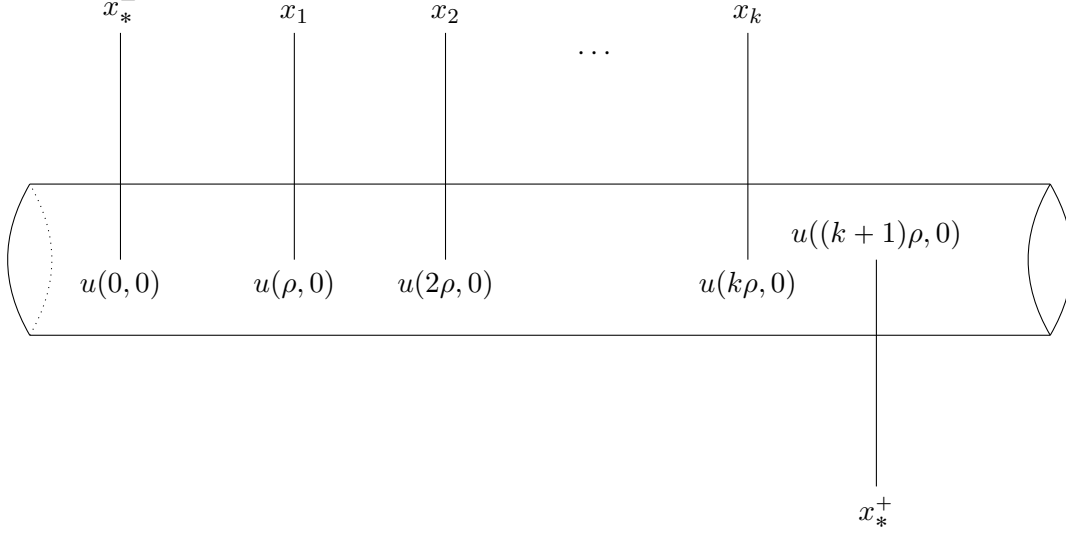
Since $\rho_n \rightarrow \rho^*$, breaking is only possible near the “ends” of the cylinder. There are two possibilities, breaking at $+\infty$ or at $-\infty$. At $-\infty$, we have to break on a critical point for $h(r)$ with action in $[A, c(B - A) + A]$ since the homotopy of Hamiltonian functions is monotone. This is impossible as the only such point is the asymptotic orbit γ_- . The argument excluding breaking at $+\infty$ is the same using the properties of $l(r)$ and the orbit γ_+ .

Bubbling is prevented since the energy of all elements $(\rho, u) \in \mathcal{M}$ curve is less than 1. Indeed, $E(u) \leq \mathcal{A}(\gamma_+) - \mathcal{A}(\gamma_-) < 1$ by construction. Therefore there is not enough energy for bubbling-off of holomorphic spheres since on $\pi_2(E) \cong \pi_2(M)$ we have $\Omega(\pi_2(E)) = \omega(\pi_2(M)) \subset \mathbb{Z}$ due to the assumption $[\omega] \in H^2(M, \mathbb{Z})$.

This shows that under the assumption that ρ stays bounded for all sequences in \mathfrak{M} , the moduli space \mathfrak{M} is compact.

By construction, the parametrized moduli space \mathfrak{M} has only one boundary component $\partial\mathfrak{M} = \mathfrak{M}|_{\rho=0} = \mathcal{M}$ which, as shown above, is Fredholm regular. By compactness, \mathfrak{M} is still Fredholm regular for small values of ρ . Using an abstract perturbation argument as in [AH16], see also [HWZ14, Theorems 5.5 and 5.13], we can define a perturbed moduli space $\widetilde{\mathfrak{M}}$, which is a smooth, 1-dimensional compact manifold and agrees with \mathfrak{M} near $\rho = 0$, where \mathfrak{M} is already Fredholm regular.

In particular, then $\widetilde{\mathfrak{M}}$ is a compact, 1-dimensional manifold with only one boundary component. As this cannot exist, we have shown that the assumption of in the beginning of the proof is wrong and there exists a sequence $\{(\rho_n, u_n)\}_{n \in \mathbb{N}}$ of elements in \mathfrak{M} with $\rho_n \rightarrow \infty$. \square

FIGURE 3. Elements in \mathfrak{M}

As the last step in this section, we also define moduli spaces for bounded values of ρ . Namely, we set

$$\mathfrak{M}_\rho(x_1, \dots, x_k, x_*^-, x_*^+) = \{u \mid (u, \rho) \in \mathfrak{M}\}$$

and

$$\mathfrak{M}[0, \rho] = \{u \mid (u, \sigma) \in \mathfrak{M} \ \forall \ \sigma \in [0, \rho]\}.$$

As in [AH16], also these moduli spaces can be perturbed to be smooth compact manifolds $\widetilde{\mathfrak{M}}_\rho$ for $\rho \in \mathbb{N}$ by an abstract perturbation argument, cf. [HWZ14, Theorems 5.5 and 5.13]. Moreover, as described above, $\mathfrak{M}_0 = \partial\mathfrak{M} = \mathcal{M}$ is already Fredholm regular and the perturbations can be done leaving \mathfrak{M}_ρ untouched for small ρ . Then we can also perturb the moduli spaces $\mathfrak{M}[0, \rho]$ for $\rho \in \mathbb{N}$ keeping the ends fixed to get smooth manifolds $\widetilde{\mathfrak{M}}[0, \rho]$.

3.2. Finding critical points of the action functional. The next step is to use the above moduli spaces to construct cohomology operations. It is rather standard, cf. [AH16, Sch93], that the projection of \mathfrak{M}_0 to M defines the cup product on M by

$$\begin{aligned} \theta_0: \text{CM}^*(f_1) \otimes \dots \otimes \text{CM}^*(f_k) \otimes \text{CM}_*(f_*) &\rightarrow \text{CM}_*(f_*) \\ x_1 \otimes \dots \otimes x_k \otimes x_*^- &\mapsto \sum_{x_*^+ \in \text{Crit}(f_*)} \#_{2pr_M} \widetilde{\mathfrak{M}}_0(x_1, \dots, x_k, x_*^-, x_*^+) \cdot x_*^+. \end{aligned} \quad (3.4)$$

Here, we use Morse homology and cohomology with coefficients in $\mathbb{Z}/2\mathbb{Z}$. Observe that the functions f_i and f_* are defined on M and for $\rho = 0$, the projection is a standard Morse bouquet as the cylinder projects to a point. Thus all homology and cohomology groups above can be identified with the singular homology and cohomology of M . Then the left hand side corresponds to the homology class

$$([x_1] \cup \dots \cup [x_k]) \cap [x_*^-]. \quad (3.5)$$

Furthermore, as $k = \text{cuplength}(M)$, we can choose generic Morse functions such that there are critical points x_*^\pm and x_1, \dots, x_k such that this product is non-zero. In particular, this shows that the moduli space $pr_M \widetilde{\mathfrak{M}}_0(x_1, \dots, x_k, x_*^-, x_*^+)$ is non empty and therefore, we also have

$$\widetilde{\mathfrak{M}}_0(x_1, \dots, x_k, x_*^-, x_*^+) \neq \emptyset. \quad (3.6)$$

As the next step, we define cohomology operations depending on $\rho \in \mathbb{N}$ by

$$\begin{aligned} \theta_\rho: \text{CM}^*(f_1) \otimes \dots \otimes \text{CM}^*(f_k) \otimes \text{CM}_*(f_*) &\rightarrow \text{CM}_*(f_*) \\ x_1 \otimes \dots \otimes x_k \otimes x_*^- &\mapsto \sum_{x_*^+ \in \text{Crit}(f_*)} \#_2 \widetilde{\mathfrak{M}}_\rho(x_1, \dots, x_k, x_*^-, x_*^+) \cdot x_*^+. \end{aligned} \quad (3.7)$$

As in [AH16, AM10], these operations are chain homotopy equivalent to θ_0 using the moduli spaces $\widetilde{\mathfrak{M}}[0, \rho]$ to define the chain homotopy. In particular, this shows that for all $n \in \mathbb{N}$, there are generic Morse functions f_i and f_* , possibly depending on n , with critical points x_i and x_*^\pm such that

$$\widetilde{\mathfrak{M}}_n(x_1, \dots, x_k, x_*^-, x_*^+) \neq \emptyset. \quad (3.8)$$

This implies that also

$$\mathfrak{M}_n(x_1, \dots, x_k, x_*^-, x_*^+) \neq \emptyset \quad (3.9)$$

as otherwise, also a small perturbation would be empty.

We now run the C_{loc}^∞ compactness k times by centering ourselves at each $l\rho_n$ for $l = 1, \dots, k$. This means that we choose $u_n \in \mathfrak{M}_n(x_1, \dots, x_k, x_*^-, x_*^+)$ and consider the sequences

$$u_{n,l}(s, t) = u_n(s + nl, t). \quad (3.10)$$

As in [AH16, AM10], these sequences converge to a broken Floer trajectory and we find $k + 1$ critical points $(\gamma_i, \overline{\gamma_i})$ such that

$$\mathcal{A}(\gamma_i, \overline{\gamma_i}) > \mathcal{A}(\gamma_{i+1}, \overline{\gamma_{i+1}}).$$

The inequalities are strict by generic choice of Morse functions. Indeed, if there were an equality, the corresponding Floer trajectory would have zero energy and thus be constant. But in this case, one of the gradient trajectories η_i of f_i would go through $\gamma_i(0)$. This can easily be prevented by a generic choice of the Morse functions. Namely, we assumed that there are only finitely many critical points of the action functional for $h\left(\frac{r}{f(x)}\right)$ and these finitely many points can be avoided by all stable manifolds of non-maximal critical points of the Morse functions f_i .

4. PROOFS OF COROLLARIES

In this section, we finally prove the statements about Reeb dynamics implied by the Theorem. In detail, we first prove Corollary 1.3 by showing that the critical point of the Hamiltonian function action functional found above correspond to closed Reeb orbits. Then we consider the special case of starshaped hypersurfaces in \mathbb{R}^{2n} . Here, known bounds for the lengths of closed Reeb orbits yield Corollary 1.5. We prove both corollaries for completeness, even though the key points of the proofs are known facts in contact dynamics.

4.1. Closed Reeb orbits. To prove Corollary 1.3, we need to show that the critical points of the action functional found above actually are closed Reeb orbits and that the action bounds given by the pinching condition excludes multiplicities in the absence of a short orbit.

For simplicity of notation, we denote our Hamiltonian function by $h_f(x, r) := h\left(\frac{r}{f(x)}\right)$ and let V_f the vector field on Σ defined by

$$\alpha(V_f) = 0, \quad df(R)\alpha - df = d\alpha(V_f, \cdot), \quad (4.1)$$

where α is the standard contact form on Σ , i.e., V is contained in the contact distribution, where the second equation uniquely defines the vector field. Furthermore, we denote by Σ_f the contact manifold defined by points of the form $(x, f(x)) \in E$ with the contact form $\alpha_f = f\alpha$.

The first step towards the proof of the corollary is to compute the Hamiltonian vector field of h_f and show that 1-periodic Hamiltonian function orbits correspond to closed Reeb orbits.

Lemma 4.1. The Hamiltonian vector field X_{h_f} of h_f is given by

$$X_{h_f}(x, r) = \frac{h'\left(\frac{r}{f(x)}\right)}{f(x)^2} \left(f(x)R(x) - V_f(x) + r df(x)[R(x)]\partial_r \right). \quad (4.2)$$

If $\gamma(t) = (x(t), r(t))$ is a 1-periodic orbit of X_{h_f} then $r(t) = cf(x(t))$ for some constant c and we define the curves $z(t) := x(t/h'(c))$ and $\zeta(t) := (z(t), f(z(t)))$. With these definitions, ζ is a Reeb orbit of R_f of period $h'(c)$, where R_f is the Reeb vector field on Σ_f defined by α_f .

PROOF. The formula for the Hamiltonian vector field is checked by computing dh_f and plugging X_{h_f} into $\omega = d(r\alpha)$. This definition uses the natural splitting of the tangent space into the radial component, the Reeb direction and the contact distribution. For the 1-form dh_f , we have

$$dh_f(x, r)[v + a\partial_r] = h' \left(\frac{r}{f(x)} \right) \left(\frac{a}{f(x)} - \frac{r}{f(x)^2} df(x)[v] \right),$$

where v is a tangent vector to Σ and $a \in \mathbb{R}$. With $(\hat{x}, \hat{r}) = v + a\partial_r$, we now compute $i_{X_{h_f}}\omega(\hat{x}, \hat{r})$ using the expression for the Hamiltonian vector field as stated.

$$\begin{aligned} d(r\alpha)(X_{h_f}(x, r), (\hat{x}, \hat{r})) &= (dr \wedge \alpha + r d\alpha)(X_{h_f}(x, r), v + a\partial_r) \\ &= \frac{h'\left(\frac{r}{f(x)}\right)}{f(x)^2} (dr \wedge \alpha + r d\alpha) \left(f(x)R(x) - V_f(x) - r df(x)[R(x)]\partial_r, v + a\partial_r \right) \\ &= \frac{h'\left(\frac{r}{f(x)}\right)}{f(x)^2} \left(r df(x)[R(x)]\alpha(v) - f(x)a - r d\alpha(V_f(x), v) \right) \\ &\stackrel{(*)}{=} \frac{h'\left(\frac{r}{f(x)}\right)}{f(x)^2} \left(r df(x)[R(x)]\alpha(v) - f(x)a - r df(x)[R(x)]\alpha(v) + r df(x)[v] \right) \\ &= \frac{h'\left(\frac{r}{f(x)}\right)}{f(x)^2} (-f(x)a + r df(x)[v]) \\ &= -dh_f(x, r)[v + a\partial_r], \end{aligned}$$

where $(*)$ uses the second equation in (4.1). This shows that (4.1) indeed is the Hamiltonian vector field.

Since h_f is autonomous, the fact that γ is a 1-periodic orbit of X_{h_f} implies that $h_f(\gamma(t))$ is constant. Thus if $\gamma(t) = (x(t), r(t))$ then $r(t)/f(x(t))$ is constant, since h is strictly increasing. Thus $\gamma(t) = (x(t), cf(x(t)))$ for some constant c .

Set $z(t) := x(t/h'(c))$ and $\zeta(t) = (z(t), f(z(t)))$. Then $\zeta(t) \in \Sigma_f$ for all t and we claim that ζ is a closed Reeb orbit of R_f . For this, we compute

$$\dot{z}(t) = \frac{1}{h'(c)} \dot{x}(t/h'(c)) = \frac{1}{f(z(t))} R(z(t)) - \frac{1}{f(z(t))^2} V_f(z(t)) \quad (4.3)$$

from (4.2). Thus to complete the proof it suffices to show that, writing

$$R_f(x, f(x)) = U(x) + df(x)[U(x)]\partial_r,$$

one has

$$U(x) = \frac{1}{f(x)} R(x) - \frac{1}{f(x)^2} V_f(x). \quad (4.4)$$

Note that the above form of R_f is determined by the fact that Σ_f can be written as a radial graph over Σ . It therefore suffices to determine U . Clearly with U given as in (4.4) one has $\alpha_\Sigma(R_\Sigma) = 1$. Now

$$\begin{aligned} d\alpha_\Sigma(R_\Sigma, \cdot) &= (df \wedge \alpha + f d\alpha) \left(\frac{1}{f(x)} R(x) - \frac{1}{f(x)^2} V_f(x), \cdot \right) \\ &= \frac{1}{f} df(R)\alpha - \frac{1}{f} df - \frac{1}{f^2} df(V_f)\alpha + \frac{1}{f^2} \alpha(V_f)df - \frac{1}{f} d\alpha(V_f, \cdot) \\ &= -\frac{1}{f^2} df(V_f) = 0, \end{aligned}$$

where we used both equations in (4.1) again. Furthermore, we used that $df(V_f) = 0$ which can be seen by feeding V_f to both sides of the second equation of (4.1). \square

To prove Corollary 1.3, we study now the Hamiltonian action functional. We denote by $\mathcal{A}_{h_f} : \mathcal{L}(M) \rightarrow \mathbb{R}$ the standard Hamiltonian function action functional

$$\mathcal{A}_{h_f}(\gamma) := \int_{S^1} \gamma^*(r\alpha) - \int_{S^1} h_f(\gamma(t)) dt$$

Corollary 4.2. *The critical points of \mathcal{A}_{h_f} correspond to closed orbits of R_Σ . Namely, if $\gamma(t)$ is a critical point of \mathcal{A}_{h_f} , then writing $\gamma(t) = (x(t), r(t))$, one has $r(t) = cf(x(t))$ for some constant c , and if $z(t) := x(t/h'(c))$ and $\zeta(t) := (z(t), f(z(t)))$ then ζ is a closed orbit of R_Σ of period $h'(c)$. Moreover,*

$$\mathcal{A}_{h_f}(\gamma) = ch'(c) - h(c). \quad (4.5)$$

PROOF. Critical points of \mathcal{A}_{h_f} are 1-periodic orbits of X_{h_f} and the action value follows immediately. The remaining parts of the claim have been proved in the previous lemma. \square

Now Corollary 1.3 follows immediately from this. As the closed orbits we found are ordered by action, the last one cannot be a multiple of the first one by the pinching condition and the bound on the slope of the Hamiltonian function h . Thus either there is a Reeb orbit such that two of the ones found by the Theorem are both covers of this short one. If this is not the case, the $k+1$ orbits found above are geometrically distinct and we have at least $k+1$ closed Reeb orbits.

4.2. Starshaped hypersurfaces in \mathbb{R}^{2n} . In this last part, we study the particular case of \mathbb{R}^{2n} , where we have concrete bounds for the length of Reeb orbits on starshaped hypersurfaces. Note that there is a change in notation in this section to match the “standard” notation used for this theorem in the literature. In particular, we do not need to use the language of the line bundle over Σ in this setting and for a given starshaped hypersurface, we do not use the defining function f any more.

Therefore, Σ will now denote the starshaped hypersurface of interest, which was denoted by Σ_f above. Similarly, we now denote by α the usual contact form on Σ , which was α_f before. In computations, we let α_x denote the form at the point $x \in \Sigma$. We reprove below the relation between the largest radius R_1 of a sphere contained in Σ and the action of closed Reeb orbits on Σ . Together with Corollary 1.3, this yields the desired Corollary 1.5.

Lemma 4.3. Let $\gamma : [0, T] \rightarrow \Sigma$ be a simple T -periodic Reeb orbit on $\Sigma \subset \mathbb{R}^{2n}$ such that the largest sphere contained in the domain bounded by Σ has radius R_1 . Assume moreover that for all $x \in \Sigma$, we have $T_x \Sigma \cap B_{R_1}(x_0) = \emptyset$. This condition is weaker than convexity which is also a common condition in similar settings. Then we have

$$T \geq \pi R_1^2.$$

Remark 4.4. The assumption that $T_x \Sigma \cap B_{R_1}(x_0) = \emptyset$ for all $x \in \Sigma$ can be reformulated as

$$\langle \nu_\Sigma(z), z \rangle > R_1, \quad \forall z \in \Sigma \quad (4.6)$$

where $\nu_\Sigma(z)$ is the exterior normal vector of Σ at point z and $\langle \cdot, \cdot \rangle$ denotes the Euclidean scalar product on \mathbb{R}^{2n} .

PROOF. We follow a similar argument in [BLMR85]. Let $\gamma : [0, T] \rightarrow \Sigma$ be a simple periodic Reeb orbit. We first compute a bound for T in terms of the Reeb vector field. The main ingredient is the special form of the contact form which is given as $\alpha_x(X_x) = \frac{1}{2} \langle X_x, Jx \rangle$.

Writing $\bar{\gamma}(t) := \gamma(t) - \int_0^T \gamma(t) dt$, we compute

$$\begin{aligned} 2T &= \int_0^T \alpha_{\gamma(t)}(\dot{\gamma}(t)) dt \\ &= \int_0^T \langle \dot{\gamma}(t), J\gamma(t) \rangle dt \\ &= \int_0^T \langle \dot{\gamma}(t), J\bar{\gamma}(t) \rangle dt \\ &\leq \|\dot{\gamma}\|_{L^2} \|\bar{\gamma}\|_{L^2} \\ &\leq \|\dot{\gamma}\|_{L^2}^2 \frac{T}{2\pi} \\ &= \frac{T}{2\pi} \int_0^T \|\dot{\gamma}(t)\|^2 dt \\ &= \frac{T}{2\pi} \int_0^T \|(R_\alpha)_{\gamma(t)}\|^2 dt, \end{aligned} \quad (4.7)$$

where we use Wirtinger’s inequality to get the second inequality.

For any point x in Σ , the norm of the Reeb vector field is bounded by $\|(R_\alpha)_x\| \leq \frac{2}{R_1}$. Indeed, we have

$$\iota(J\nu_\Sigma)d\alpha(Y) = \omega(J\nu_\Sigma, Y) = -\langle \nu_\Sigma, Y \rangle = 0 \quad (4.8)$$

for all $Y \in T\Sigma$. Therefore, R_α is proportional to $J\nu_\Sigma$ and we have $R_\alpha = cJ\nu_\Sigma$ with $|c| = \|R_\alpha\|$.

On the other hand, we also use the second defining equation for the Reeb vector field to get

$$1 = \alpha_x(R_{\alpha_x}) = \frac{1}{2} \langle c_x J\nu_\Sigma(x), Jx \rangle = \frac{c_x}{2} \langle \nu_\Sigma(x), x \rangle \quad (4.9)$$

and therefore, we find $c_x = \frac{2}{\langle \nu_\Sigma(x), x \rangle} \leq \frac{2}{R_1}$.

This gives rise to an upper bound for the last line in (4.7). Namely, we have

$$\frac{T}{2\pi} \int_0^T \|(R_\alpha)_{\gamma(t)}\|^2 dt \leq \frac{4}{R_1^2} T \frac{T}{2\pi} \quad (4.10)$$

and in total, we have shown that $2T \leq 2T \frac{T}{\pi R_1^2}$ which implies the lemma. \square

Finally, using this lemma, we can prove Corollary 1.5 to obtain the theorem by Ekeland-Lasry as a cuplength estimate.

Proof of Corollary 1.5. We now view the $2n - 1$ sphere as the boundary of the ball blown-up at the origin. This point of view gives the sphere as a circle bundle in the tautological complex line bundle $\mathcal{O}(-1)$ over \mathbb{CP}^{n-1} . Note that the Reeb dynamics is unaffected by this consideration. Theorem 1.1 gives us the existence of n periodic Reeb orbits on the sphere whose action is “pinched”

$$\pi R_1^2 < \mathcal{A}(\gamma_1) < \dots < \mathcal{A}(\gamma_n) < \pi R_2^2.$$

The condition $R_2^2 < 2R_1^2$ corresponds to the above condition that $R_0 < 2$ and therefore, these n Reeb orbits cannot be iterates of one another. The lower bound on the period of closed Reeb orbits on Σ given by Lemma 4.3 above shows that they can also not be iterates of a short orbit. Thus we have n simple periodic Reeb orbit. \square

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PETER ALBERS, MATHEMATISCHES INSTITUT, WESTFÄLISCHE WILHELMS-UNIVERSITÄT MÜNSTER
E-mail address: `peter.albers@wwu.de`

JEAN GUTT, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF GEORGIA
E-mail address: `jpgutt@uga.edu`

DORIS HEIN, MATHEMATISCHES INSTITUT, ALBERT-LUDWIGS-UNIVERSITÄT FREIBURG
E-mail address: `doris.hein@math.uni-freiburg.de`